On some properties of the Landau kinetic equation

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Introduction

Kinetic equations for plasmas

$f_i(x, v, t),\ i = 1, .., n$- distribution functions;

$x \in \mathbb{R}^3$ - position, $v \in \mathbb{R}^3$ -velocity, $t \in \mathbb{R}_+$ -time

Evolution equations: $D_i f_i = \sum_{j=1}^{n} Q_{ij}^L(f_i, f_j),$ where

$$D_i = \partial_t + v \cdot \partial_x + \frac{e_i}{m_i}\left(E + \frac{1}{c} v \times B\right) \cdot \partial_v, \qquad i = 1, .., n$$

$E(x, t)$ and $B(x, t)$ are electric and magnetic vector fields

(external + self-consistent fields)

`Standard splitting on` $[t_0, t_0 + \Delta t]$:

- (1) $D_i f_i = 0$ $\longleftarrow$ Vlasov equations
- (2) $\partial_t f_i = \sum_{j=1}^{n} Q_{ij}^L(f_i, f_j)$ $\longleftarrow$ Landau (LFP) equations

Particle Methods: $\begin{cases} (1) & \text{PiC for Vlasov} \\ (2) & \text{DSMC for Landau} \end{cases}$

Landau Equation (LE)

Landau (1936) generalized Boltzmann equation to the case of Coulomb interaction

$$U_{ij} = \frac{e_i e_j}{|x_i - x_j|} \quad \text{-} \quad \text{interparticle potential}$$

Rough idea:

Consider a modified potential $\widetilde{U}_{ij} = U_{ij} \exp(-r_{ij}/r_D)$, with the Debye radius r_D, and find the leading asymptotic term of the Boltzmann collision integral, as $r_D \to \infty$.

The result reads

$$\frac{\partial f_i(\mathbf{v},t)}{\partial t} = 2\pi L \sum_{j=1}^{n} \frac{e_i^2\, e_j^2}{m_i^2} \frac{\partial}{\partial v_\alpha} \int_{\mathbb{R}^3} d\mathbf{w}\, R_{\alpha\beta}(\mathbf{v}-\mathbf{w}) \left(\frac{\partial}{\partial v_\beta} - \frac{m_i}{m_j} \frac{\partial}{\partial w_\beta} \right) f_i(\mathbf{v}) f_j(\mathbf{w}),$$

where

$$R_{\alpha\beta}(\mathbf{u}) = \frac{u^2 \delta_{\alpha\beta} - u_\alpha u_\beta}{u^3}, \qquad \alpha,\beta = 1,2,3, \qquad i,j = 1,...,n,$$

$L = \log(r_D/r_0)$ - Coulomb logarithm

LE in Fokker-Planck form

Rosenbluth, MacDonald, and Judd (1957) re-discovered LE by postulating it in FP form.

The results reads

$$\frac{1}{4\pi L}\frac{\partial f_i}{\partial t} = \frac{\partial}{\partial v_\alpha}\left\{-f_i\frac{\partial h_i}{\partial v_\alpha} \quad + \frac{1}{2}\frac{\partial}{\partial v_\beta}\left(f_i\frac{\partial^2 g_i}{\partial v_\alpha \partial v_\beta}\right)\right\},$$

h_i and g_i are called **Rosenbluth potentials**

$$h_i = \sum_{j=1}^{n} K_{ij} \int_{\mathbb{R}^3} d\mathbf{w} f_j(\mathbf{w}, t) |\ \mathbf{v} - \mathbf{w}\ |^{-1}, \quad g_i = \sum_{j=1}^{n} K_{ij}\frac{m_j}{m_i}\int_{\mathbb{R}^3} d\mathbf{w} f_j(\mathbf{w}, t) |\ \mathbf{v} - \mathbf{w}\ |,$$

$$K_{ij} = \frac{e_i^2 e_j^2}{m_i m_j}, \quad i = 1, .., n$$

This form of LE is very useful for regular (deterministic) numerical methods.

Introduction. Key references.

Landau 1936 (Coulomb forces)

Bogolyubov 1946 (validation)

Rosenbluth et. al 1957 (Fokker-Planck form)

Balescu 1961 and Lenard 1961 (B-L equation)

A lot of papers by physicists in 60s - 70s...

Mathematical works

(A) More detailed study of connection with Boltzmann equation:

B. (1975), Arsen'ev (1989), Degond and Lucquin-Desreux (1992), Desvillettes (1992),...

(B) Existence and uniqueness theorems

Villani (1998), Desvillettes and Villani (2000), Desvillettes (2014), Guo (2002), Strain and Guo (2008),...

(C) Numerical works

(a) finite difference schemes: Rosenbluth et al. (1957), ..., B., Chuyanov, Potapenko, Pekker in 70s-80s, Degond, Buet, Cordier in 90s,...

(b) spectral methods: Toscani, Russo, Pareschi, Filbet (1990s - 2000s), Gamba and Haack (2014),...

(c) DSMC methods: Takizuka-Abe(1977), Nanbu (1997), B. and Nanbu (2000), Caflisch et al. (2005,...), B. and Potapenko (2012),...

I. The Landau equation and its connection with the Boltzmann equation

Notation: $f(v,t)$ - distribution function ($v \in \mathbb{R}^3,\ t \geq 0$).

Spatially homogeneous Landau equation:

$$\frac{\partial f}{\partial t} = B \frac{\partial}{\partial v_i} \int\limits_{\mathbb{R}^3} dw \ \frac{(u^2 \delta_{ij} - u_i u_j)}{u^3} \left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial w_j} \right) f(v) f(w),$$

$u = v - w, \quad i, j = 1, 2, 3.$

The constant B depends on intermolecular potential $U(r)$, where $r > 0$ is the distance between two interacting particles,

$$B = \frac{1}{8\pi} \int\limits_0^\infty dr\, r^3\, \hat{U}(r)^2, \quad \hat{U}(|k|) = \int\limits_{\mathbb{R}^3} dx U(|x|) e^{ik\cdot x}.$$

Generalized Landau collision integral

$$Q_L(f,f) = \frac{1}{8}\frac{\partial}{\partial v_i}\int\limits_{\mathbb{R}^3} dw\ \Phi(|u|)\ R_{ij}(u)\left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial w_j}\right) f(v)f(w),$$

$$R_{ij}(u) = (|u|^2\delta_{ij} - u_i u_j), \quad \Phi(|u|) = 2\pi|u|\int\limits_{-1}^{1} d\mu\sigma(|u|,\mu)(1-\mu),$$

$(\sigma(|u|,\cos\theta)$ - a differential cross-section of scattering at the angle $0 \le \theta \le \pi)$

corresponds to the Boltzmann integral

$$Q_B(f,f) = \int\limits_{\mathbb{R}^3\times\mathbb{S}^2} dw d\omega g\left(|u|, \frac{u\cdot\omega}{|u|}\right)[f(v')f(w') - f(v)f(w)],\ \omega \in \mathbb{S}^2$$

where

$$g(|u|,\mu) = |u|\sigma(|u|,\mu), \quad v' = \frac{1}{2}(v+w+|u|\omega), \quad w' = \frac{1}{2}(v+w-|u|\omega).$$

Grazing collision limit:

Take a family of kernels $g_\varepsilon(|u|, \mu)$ such that

$$\lim_{\varepsilon\to 0} 2\pi \int_{-1}^{1} d\mu\, g_\varepsilon(|u|, \mu)(1-\mu)^k = \begin{cases} \Phi(|u|) & \text{if} \quad k = 1, \\ 0 \quad \text{if} & k > 1. \end{cases}$$

Then formally: $\quad \Delta Q = Q_B - Q_L \;\to\; 0 \quad \text{as} \quad \varepsilon \;\to\; 0.$

Most typical kernel

$$g_\varepsilon(|u|, \mu) = |u|^\gamma g_\varepsilon(\mu), \quad \text{where} \quad \gamma = 1 - \frac{4}{n},\; n \geq 1.$$

It formally corresponds to the potential $U(r) \sim r^{-n}$, $n \geq 1$. Then we obtain $\Phi(|u|) = C_\gamma |u|^\gamma, \quad -3 \leq \gamma < 1.$

Remark. Only $\gamma = -3$ (”true” LE) is justified from physics!

Quasi-Maxwellian approximation.

We assume that $g_{tot,\varepsilon}(|u|) = 2\pi \int\limits_{-1}^{1} d\mu g_\varepsilon(|u|, \mu) = \frac{1}{\varepsilon}$,

e.g. $g_\varepsilon(|u|, \mu) = \frac{1}{2\pi\varepsilon}\delta[1 - \mu - \min(\varepsilon\Phi(|u|), 2)]$.

Then the Boltzmann integral reads $Q_B(f, f) = Q_\varepsilon{}^+(f, f) - \frac{\rho(f)}{\varepsilon} f$,

$$\rho(f) = \int\limits_{\mathbb{R}^3} dv\, f(v), \quad Q_\varepsilon{}^+(f_1, f_2) = \int\limits_{\mathbb{R}^3 \times \mathbb{S}^2} dw d\omega\, g_\varepsilon\left(|u|, \frac{u \cdot \omega}{|u|}\right) f_1(v') f_2(w').$$

Remark. Known DSMC methods for LE are implicitly based on quasi-Maxwellian approximation.

II. Cauchy problem and Wild sum

$$f_t^{(\varepsilon)} + \frac{1}{\varepsilon} f^{(\varepsilon)} = Q_\varepsilon{}^+\left(f^{(\varepsilon)}, f^{(\varepsilon)}\right), \quad f^{(\varepsilon)}|_{t=0} = f_0(v) \geq 0.$$

Wild sum

$$f^{(\varepsilon)}(v,t) = \sum_{n=0}^{\infty} e^{-\frac{t}{\varepsilon}} \left(1 - e^{-\frac{t}{\varepsilon}}\right)^n f_n^{(\varepsilon)}(v),$$

where

$$f_0^{(\varepsilon)} = f_0, \quad f_{n+1}^{(\varepsilon)} = \frac{\varepsilon}{n+1} \sum_{k=0}^{n} Q_\varepsilon{}^+\left(f_k^{(\varepsilon)}, f_{n-k}^{(\varepsilon)}\right).$$

We expect that $f^{(\varepsilon)}(v,t) \to f(v,t)$, as $\varepsilon \to 0$, where $f(v,t)$ solves the problem

$$f_t = Q_L(f,f), \quad f|_{t=0} = f_0(v).$$

Radial solution If $f(v,t) = \tilde{f}(\tilde{v},t)$, where $\tilde{v} = |v|$.

Omitting tildes we obtain ($v \geq 0$ is the scalar variable here and below!)

$$Q_L(f,f) = \frac{1}{v^2}\frac{\partial}{\partial v}\int_0^\infty dw\, wK(v,w)\,\left(\frac{1}{v}\frac{\partial}{\partial v} - \frac{1}{w}\frac{\partial}{\partial w}\right) f(v)f(w),$$

$$K(v,w) = K(w,v) = \frac{\pi}{4}\int_{-1}^{1} d\mu(1-\mu^2)\Phi\left(\sqrt{v^2+w^2-2\mu vw}\right).$$

More conventional form: $Q_L(f,f) = \dfrac{1}{v^2}\dfrac{\partial}{\partial v}\left[A(v)\dfrac{\partial f}{\partial v} + B(v)f\right]$,

$$A(v) = \frac{1}{v}\int_0^\infty dw wK(v,w)f(w), \quad B(v) = \int_0^\infty dw f(w)\frac{\partial}{\partial w}K(v,w).$$

Note that $\Phi(|u|) = c_\gamma |u|^\gamma$, $1 > \gamma \geq -3$, for power-like potentials.

In the Coulomb case $\gamma = -3$:

$$K(v,w) = \frac{1}{3}\min(v^3, w^3),$$

$$A(v) = \frac{1}{3v}\left[\int_0^v dw\, w^4 f(w) + v^3 \int_v^\infty dw\, w\, f(w)\right],$$

$$B(v) = \int_0^v dw\, w^2 f(w).$$

Then the true (radial) Landau equation reads

$$\frac{\partial f}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left[A(v)\frac{\partial f}{\partial v} + B(v) f\right], \quad f|_{t=0} = f_0(v),$$

A model of the radial Landau equation

LE for $\gamma = -3$ can be also written as

$$f_t = \frac{1}{x^\theta}\frac{\partial}{\partial x}\int\limits_0^\infty dy \min(x^{1+\theta}, y^{1+\theta})\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) f(x)f(y), \quad f|_{t=0} = f_0(x),$$

where $\theta = 1/2$. This equation can be also considered for any $\theta \in [0, 1/2]$.

It has two conservation laws and H-theorem:

$$\frac{dm_0(t)}{dt} = \frac{dm_1(t)}{dt} = 0, \quad \frac{dH(t)}{dt} \leq 0, \quad \text{where}$$

$$m_2(t) = \int\limits_0^\infty dx x^{\theta+n} f(x,t), \quad H(t) = \int\limits_0^\infty dx x^\theta f(x,t) \ln f(x,t).$$

The main properties of the equation do not depend on specific value of θ.

Instead of the "true" value $\theta = 1/2$ we can consider a model equation with $\theta = 0$.

Then we obtain $$f_t = \partial_x[D(x,t)f_x + F(x,t)f], \quad f|_{t=0} = f_0(x),$$

$$D(x,t) = \int_0^\infty dy f(y,t)\min(x,y), \quad F(x,t) = \int_0^x dy f(y,t).$$

We can reduce this equation to a usual (local) nonlinear parabolic equation for $D(x,t)$:

$$D_t = DD_{xx} - D_x^2 + D_x, \quad D|_{t=0} = \int_0^\infty dy f_0(y)\min(x,y).$$

The existence and uniqueness of a smooth solution $D(x,t)$ can be probably proved by more or less standard methods.

Note that $u = D^{-1}$ satisfies the equation $u_t = \partial_x\left(\frac{1}{u}u_x + u\right).$

III. Formation of Maxwellian tails and large time asymptotics

We consider below the radial LE with $\Phi(|u|) = c_\gamma |u|^\gamma$ (power-like potentials), $-3 \le \gamma < 1$.

Then

$$f_t = \frac{1}{v^2} \frac{\partial}{\partial v} [A_\gamma(v) f_v + B_\gamma(v) f], \quad f|_{t=0} = f_0(v),$$

where

$$A_\gamma(v) \simeq \frac{m_1}{3} v^{2+\gamma}, \quad B_\gamma(v) \simeq m_0 v^{3+\gamma}, \quad v \to \infty,$$

and

$$m_0 = \int_0^\infty dv f(v,t) v^2 = const, \quad m_1 = \int_0^\infty dv f(v,t) v^4 = const.$$

Without loss of generality we can assume that $m_0 = (4\pi)^{-1}$, $m_1 = 3m_0$.

Hence, the asymptotic form of LE reads

$$f_t = \frac{1}{v^2}\frac{\partial}{\partial v} v^{3+\gamma}\left(\frac{1}{v} f_v + f\right), \quad v \gg 1,$$

We usually assume the convergence to equilibrium, i.e. that

$$f \xrightarrow[t\to\infty]{} M(v) = (2\pi)^{-3/2} e^{-v^2/2}.$$

If $-3 \leq \gamma < -2$ (very soft potentials) then we can expect

(a) relatively slow propagation of $f(v,t)$ to the domain of large v. This kind of behavior was clearly seen in early numerical experiments for the Landau equation

&

(b) relatively fast relaxation to equilibrium for thermal velocities $v = O(1)$

Assymptotics $\mathbf{v} \to \infty$: traveling waves for the Landau equation with $-3 \le \gamma < -1$

Step 1. Substitution $f(v,t) = M(v)u(x,t), \quad x = v^\beta, \quad \beta = \dfrac{2-\gamma}{2}$

leads to equation

$$u_t + \frac{1}{p} x^{1-p} u_x = a^2 u_{xx}, \quad p = -\frac{2\gamma}{2-\gamma}, \; a^2 = \frac{(2-\gamma)^2}{4|\gamma|}, \quad \gamma < 0.$$

Step 2. Transformation $u(x,t) = \varphi(z,t), \; z = x - t^{1/p}$

$$\text{yields} \qquad \varphi_t + \frac{t^{(1-p)/p}}{p} \left[\left(1 + \frac{z}{t^{1/p}}\right)^{1-p} - 1 \right] \varphi_z = a^2 \varphi_{zz}.$$

An asymptotic equation for large $t > 0$ and bounded $z \in [-R, R]$ reads

$$\varphi_t + \frac{(1-p)z}{pt} \varphi_z = a^2 \varphi_{zz}, \quad |z| \ll t^{1/p}.$$

Note that the function $u = \varphi = 1$ corresponds to the equilibrium state.

Step 3. Self-similar solution.

Transformation $\varphi(z,t) = \psi(y), \quad y = z/\sqrt{t},$

$$\psi(y) = \frac{b}{\sqrt{\pi}} \int_y^\infty ds \exp(-b^2 s^2), \quad b^2 = \frac{2(|\gamma| - 1)}{(2 + |\gamma|)^2}, \quad \gamma < -1,$$

yields for $\gamma < -1$ the self-similar solution

$$f(v,t) \simeq M(v)F(v,t), \quad F(v,t) = \psi\left[\frac{v^\beta - v_f^\beta(t)}{\sqrt{t}}\right],$$

$$v_f(t) = t^{1/\beta p} = t^{-1/\gamma}, \quad \beta = \frac{2-\gamma}{2}, \quad p = -\frac{\gamma}{\beta}, \quad \gamma < -1,$$

The function $F(v,t)$ looks like a traveling wave with asymptotic values $F_- = 1$ on the left and $F_+ = 1$ on the right.

Stability of the front

The front of the wave is defined by equality $F[v_f(t), t] = 1/2$, then we obtain $v_f(t) = t^{-1/\gamma}$, $\gamma < -1$.

A remarkable property of the wave is the stability of its structure. Indeed the width of the front can be characterized by a quantity

$$l_f(t) = F(v_f, t)/|F_v(v_f, t)| = \left(-2F_v(v_f, t)\right)^{-1}.$$

We obtain

$$l_f^{-1}(t) = 2\psi'(0)\beta\, v_f^{\beta-1}(t)t^{-1/2} = \sqrt{\frac{2}{\pi}(|\gamma| - 1)} = const, \quad \gamma < -1,$$

i.e. the width does not depend on time. The above asymptotic solution is expected to be valid for large v and t provided $\gamma < -1$. A similar asymptotics for the Landau equation is conjectured for a class of initial data concentrated in the thermal domain.

This conjecture is in excellent agreement with numerical results. It would be interesting to find a rigorous proof.

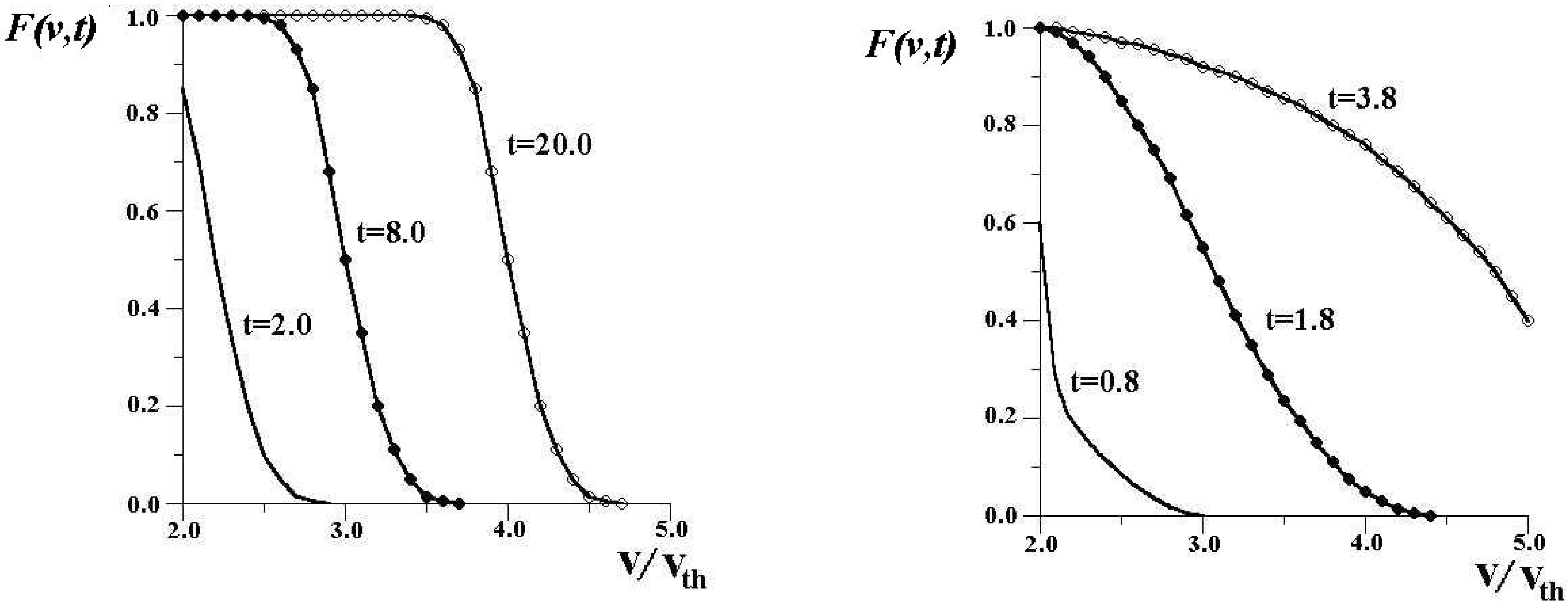

Figure 1: $\gamma = -3$ - on the left, $\gamma = -1$ - on the right

Rate of convergence to equilibrium

Let us consider the difference

$$\Delta(v,t) = M(v) - f(v,t) = M(v)\psi_1[y(v,t)],$$

$$y(v,t) = \frac{v^\beta - v_f^\beta(t)}{\sqrt{t}},\ \psi_1(y) = \frac{b}{\sqrt{\pi}} \int\limits_{-\infty}^{y} ds\, e^{-b^2 s^2}.$$

We are interested in the distance $\delta(t)$ between $f(v,t)$ and $M(v)$ given by

$$\delta(t) = \|\Delta(v,t)\| = \max_{v \geq 0} \Delta(v,t), \quad t \to \infty.$$

We can prove the estimate

$$c_1 \exp\left(-\frac{t^q}{2}\right) \leq \delta(t) \leq c_2 \exp\left(-\theta\frac{t^q}{2}\right), \quad q = \frac{2}{|\gamma|},$$

with some constant $0 < \theta < 1$.

Thus, for very soft potentials with

$$-3 \leq \gamma < -2$$

we have an explicit example of approximate solution to the Landau equation, which rate of relaxation is defined by

$$\exp(-\lambda t^q),\ 0 < q < 1.$$

The exponent

$$q = 2/|\gamma|$$

has precisely the value predicted earlier in Theorem 1 of the paper by Strain and Guo (2008).

Conclusions

1. We have discussed some some general properties of the Landau kinetic equation. In particular, the difference between ”true” Landau equation, which formally follows from classical mechanics, and ”generalized” Landau equation, which is just an interesting mathematical object, was stressed. It was shown how to approximate the Landau equation by the Wild sum. It is the so-called quasi-Maxwellian approximation related to Monte Carlo methods. This approximation can be also useful for mathematical problems.

Conclusions

2. A model equation which can be reduced to a "local" nonlinear parabolic equation was also constructed in connection with existence of the strong solution to the initial value problem.

3. The self-similar asymptotic solution to the Landau equation for large v and t was discussed in detail. The solution, earlier confirmed by numerical experiments, describes a formation of Maxwellian tails for a wide class of initial data concentrated in the thermal domain. It was shown that the corresponding rate of relaxation (fractional exponential) is in exact agreement with mathematically rigorous estimates by Strain and Guo.

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THANK YOU!......