Boundary layers for the discrete Boltzmann equation: mixtures, polyatomic molecules, chemical reactions and quantum extensions

Niclas Bernhoff

Department of Mathematics and Computer Science, Karlstad University

Karlstad, January 27 2015

Discrete Boltzmann equation (DBE)

General discrete velocity model (Gatignol (1975), Cabannes (1980)):

$$\frac{\partial F_i}{\partial t} + \xi_i \cdot \nabla_{\mathbf{x}} F_i = Q_i\left(F, F\right), \; i = 1, ..., n$$

where

$$\xi \in \mathrm{V} = \{\xi_1, ..., \xi_n\} \subset \mathbb{R}^d$$
$$F = (F_1, ..., F_n), \; F_i = F_i\left(\mathbf{x}, t\right) = F\left(\mathbf{x}, t, \xi_i\right)$$

(in applications $d = 2, 3$)

Approximation of BE with discrete velocity models (DVMs):
Bobylev, Palczewski and Schneider (1995),
Fainsilber, Kurlberg and Wennberg (2006)

Collision operator:

$$\begin{aligned} Q_i(F,F) &= \sum_{j,k,l=1}^{n} \Gamma_{ij}^{kl}\left(F_k F_l - F_i F_j\right),\ i=1,...,n \\ Q &= (Q_1,...,Q_n) \end{aligned}$$

Collision coefficients:

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0$$

with equality unless (preservation of momentum and energy)

$$\xi_i + \xi_j = \xi_k + \xi_l \text{ and } |\xi_i|^2 + |\xi_j|^2 = |\xi_k|^2 + |\xi_l|^2$$

Collision invariants:

$$\phi_i + \phi_j = \phi_k + \phi_l \text{ if } \Gamma_{ij}^{kl} \neq 0$$

Normal DVMs (only with physical collision invariants):
(Constructions: Bobylev and Cercignani 1999 (JSP), Bobylev and Vinerean 2008 (JSP) etc.):

$$\phi = (\phi_1, ..., \phi_n),\ \phi_i = a + \mathbf{b} \cdot \xi_i + c\,|\xi_i|^2$$

$$\text{constant } a, c \in \mathbb{R},\ \mathbf{b} \in \mathbb{R}^d$$

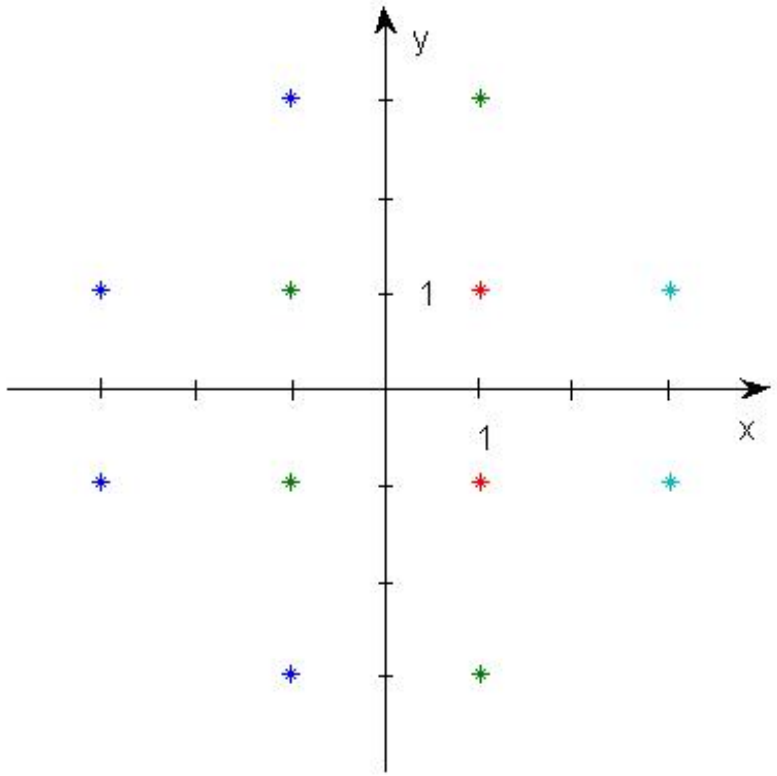
Equilibrium distributions: Maxwellian distributions/Maxwellians:

$$M = (M_1, ..., M_n),\ M_i = e^{\phi_i} = K e^{\mathbf{b}\cdot\xi_i + c|\xi_i|^2},\ \text{with } K = e^a > 0$$

$$Q(F, F) = 0 \text{ and } F_i \geq 0 \text{ if and only if } F = M$$

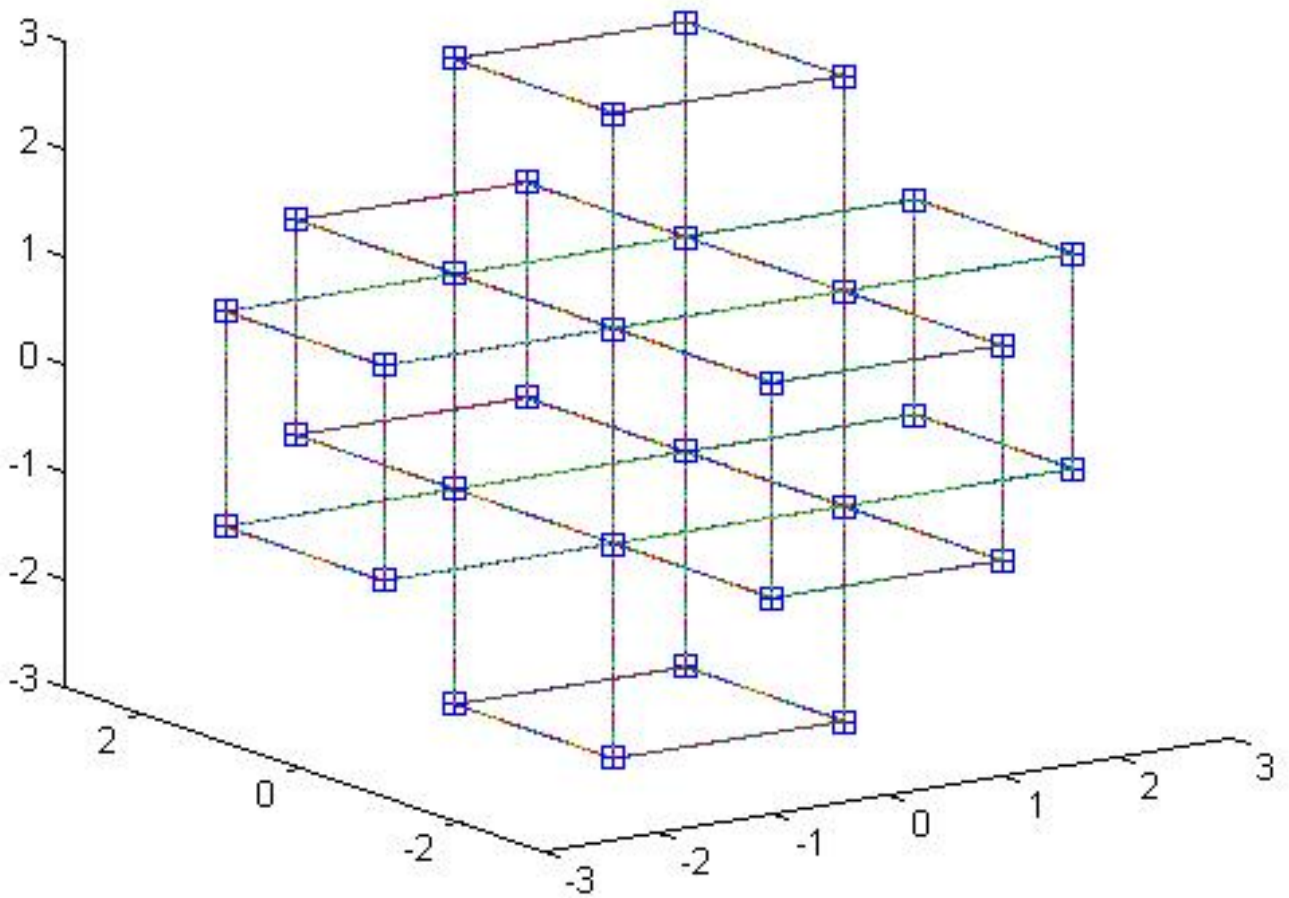
Symmetric 12-velocity model in 2d

Velocities: $(\pm 1, \pm 1)$, $(\pm 1, \pm 3)$ and $(\pm 3, \pm 1)$



Symmetric 32-velocity model in 3d

Velocities: $(\pm 1, \pm 1, \pm 1)$, $(\pm 1, \pm 1, \pm 3)$, $(\pm 1, \pm 3, \pm 1)$ and $(\pm 3, \pm 1, \pm 1)$



Linearized collision operator

$$F_i = M_i + M_i^{1/2} f_i,\ i = 1, ..., n$$

New system (of PDEs):

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_{\mathbf{x}} f + Lf = S(f)$$

Linearized collision operator ($n \times n$ matrix):

$$(Lf)_i = -2M_i^{-1/2} Q_i(M, M^{1/2} f)$$

Quadratic part:

$$S_i(f) = M_i^{-1/2} Q_i(M^{1/2} f, M^{1/2} f)$$

L is symmetric and positive semi-definite

$$\langle g, Lh\rangle = \langle Lg, h\rangle \text{ and } \langle h, Lh\rangle \geq 0,$$

$\langle \cdot, \cdot \rangle$ - Euclidean scalar product
Kernel of L:

$$N(L) = \left\{ y \mid y_i = M_i^{1/2}\phi_i,\ \phi \text{ collision invariant} \right\}$$

The nonlinear part is orthogonal to the kernel of L:

$$\langle S(f), y\rangle = 0 \text{ if } y \in N(L)$$

and, for some konstant $\widetilde{K}$:

$$|S(g) - S(h)| \leq \widetilde{K}(|g| + |h|)\,|g - h|$$

Discrete Boltzmann equation for binary mixtures

Binary mixture of the species A and B ($V^\alpha = \{\xi_1^\alpha, ..., \xi_{n^\alpha}^\alpha\} \subset \mathbb{R}^d$):

$$\begin{cases} \frac{\partial F_i^A}{\partial t} + \xi_i^A \cdot \nabla_{\mathbf{x}} F_i^A = Q_i^{AA}(F^A, F^A) + Q_i^{BA}(F^B, F^A),\ i = 1, ..., n_A \\ \frac{\partial F_j^B}{\partial t} + \xi_j^B \cdot \nabla_{\mathbf{x}} F_j^B = Q_j^{AB}(F^A, F^B) + Q_j^{BB}(F^B, F^B),\ j = 1, ..., n_B \end{cases}$$

Collision operators ($F_i^\alpha = F_i^\alpha(x) = F^\alpha(x, \xi_i^\alpha)$):

$$Q_i^{\beta\alpha}(F^\beta, F^\alpha) = \sum_{k=1}^{n_\alpha} \sum_{j,l=1}^{n_\beta} \Gamma_{ij}^{kl}(\beta, \alpha)\left(F_k^\alpha F_l^\beta - F_i^\alpha F_j^\beta\right) \text{ for } i = 1, ..., n_\alpha$$

$$\Gamma_{ij}^{kl}(\alpha, \alpha) = \Gamma_{ji}^{kl}(\alpha, \alpha) \text{ and } \Gamma_{ij}^{kl}(\alpha, \beta) = \Gamma_{kl}^{ij}(\alpha, \beta) = \Gamma_{ji}^{lk}(\beta, \alpha) \geq 0$$

with equality unless (preservation of momentum and energy)

$$m^\alpha \xi_i^\alpha + m^\beta \xi_j^\beta = m^\alpha \xi_k^\alpha + m^\beta \xi_l^\beta \text{ and}$$

$$m^\alpha \left|\xi_i^\alpha\right|^2 + m^\beta \left|\xi_j^\beta\right|^2 = m^\alpha \left|\xi_k^\alpha\right|^2 + m^\beta \left|\xi_l^\beta\right|^2$$

Collision invariants for normal discrete velocity models (DVMs): (Constructions: Bobylev and Cercignani 1999 (JSP), Bobylev and Vinerean 2008 (JSP) etc.):

$$\phi = \left(\phi^A, \phi^B\right), \text{ with } \phi^\alpha = \phi^\alpha(\xi) = a^\alpha + m^\alpha \mathbf{b} \cdot \xi + c m^\alpha \left|\xi\right|^2$$
$$\text{constant } a^A, a^B, c \in \mathbb{R},\ \mathbf{b} \in \mathbb{R}^d$$

Maxwellians:

$$M = e^\phi, \text{ i.e. } M = \left(M^A, M^B\right), \text{ with } M^\alpha = e^{\phi^\alpha}$$

where (for normal DVMs)

$$\phi = \left(\phi^A, \phi^B\right), \text{ with } \phi^\alpha = \phi^\alpha(\xi) = a^\alpha + m^\alpha \mathbf{b} \cdot \xi + c m^\alpha \left|\xi\right|^2$$

for some constant $a^A, a^B, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$

Linearized collision operator

$$F^\alpha = M^\alpha + \sqrt{M^\alpha} f^\alpha$$

New system:

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_{\mathbf{x}} f + Lf = S(f), \text{ with}$$
$$f = \left(f^A, f^B\right) \text{ and } \xi = \left(\xi^A, \xi^B\right)$$

L - symmetric and positive semi-definite matrix

$N(L) = \left\{\sqrt{M}\phi, \, \phi \text{ collision invariant}\right\}$

$S(h) \in N(L)^\perp$

For some konstant $\widetilde{K}$:

$$|S(g) - S(h)| \leq \widetilde{K}(|g| + |h|)\,|g - h|$$

DBE for mixtures - different approach

Assume now that we have s different species, labelled with $\alpha_1, ..., \alpha_s$, with the masses $m^{\alpha_1}, ..., m^{\alpha_s}$. For each species α_i we fix a set of velocity vectors $V^{\alpha_i} = \{\xi_1^{\alpha_i}, ..., \xi_{n^{\alpha_i}}^{\alpha_i}\} \subset \mathbb{R}^d$ and assign the label α_i to each velocity vector in V^{α_i}. We obtain a set of $N = n^{\alpha_1} + ... + n^{\alpha_s}$ pairs

$$\begin{aligned}\mathcal{P} &= \left\{\left(\xi_1^{\alpha_1}, \alpha_1\right), ..., \left(\xi_{n^{\alpha_1}}^{\alpha_1}, \alpha_1\right), ..., \left(\xi_1^{\alpha_s}, \alpha_s\right), ..., \left(\xi_{n^{\alpha_s}}^{\alpha_s}, \alpha_s\right)\right\} \\ &= \left\{\left(\mathbf{p}_1, \alpha(1)\right), ..., \left(\mathbf{p}_N, \alpha(N)\right)\right\}, \text{ with } N = n^{\alpha_1} + ... + n^{\alpha_s}.\end{aligned}$$

Collision coefficients:

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0$$

with equality unless

$$\{\alpha(i), \alpha(j)\} = \{\alpha(k), \alpha(l)\}$$

$$m^{\alpha(i)}\mathbf{p}_i + m^{\alpha(j)}\mathbf{p}_j = m^{\alpha(k)}\mathbf{p}_k + m^{\alpha(l)}\mathbf{p}_l \text{ and}$$

$$m^{\alpha(i)}\left|\mathbf{p}_i\right|^2 + m^{\alpha(j)}\left|\mathbf{p}_j\right|^2 = m^{\alpha(k)}\left|\mathbf{p}_k\right|^2 + m^{\alpha(l)}\left|\mathbf{p}_l\right|^2$$

$d+s+1$ collision invariants for normal DVMs:

$$\phi = \left(\phi^{\alpha_1}, ..., \phi^{\alpha_s}\right), \text{ with } \phi^{\alpha_i} = \phi^{\alpha_i}(\mathbf{p}) = a^{\alpha_i} + m^{\alpha_i}\mathbf{b}\cdot\mathbf{p} + cm^{\alpha_i}\left|\mathbf{p}\right|^2$$

$$\text{constant } a^{\alpha_1}, ..., a^{\alpha_s}, c \in \mathbb{R},\ \mathbf{b} \in \mathbb{R}^d$$

Maxwellians:

$$M = e^{\phi}, \text{ i.e. } M = \left(M^{\alpha_1}, ..., M^{\alpha_s}\right), \text{ with } M^{\alpha_i} = e^{\phi^{\alpha_i}}$$

Algorithm for construction of supernormal DVMs for mixtures

A supernormal DVM for mixtures is a DVM such that the restriction to each collection of velocity sets is normal, which is equivalent to the fact that the restriction to each pair of velocity sets is supernormal

Algorithm:
1) Choose a set of velocities V^{α_1} such that it corresponds to a normal model for single species
2) Iteration step. For $i = 2, \ldots, s$
Choose a normal set of velocities V^{α_i} such that, it together with each of $V^{\alpha_1}, \ldots, V^{\alpha_{i-1}}$ corresponds to a supernormal model for binary mixtures

In each case we need to have $d+1$ linearly independent (also with respect to the collisions inside the two species) collisions between the two species

Example- construction of normal models for a mixture of s species

We start with a set of velocities V, which includes the 8 velocities $(\pm 1, \pm 1), (3, \pm 1), (1, 3), (3, 3)$ for $d = 2$ or the 12 velocities $(\pm 1, \pm 1, \pm 1), (3, \pm 1, 1), (1, 3, 1), (3, 3, 1)$ for $d = 3$, and is normal

Extensions to larger normal models (of any finite size) by one-extensions

Smallest "symmetric" normal extensions: 16-velocity model $(\pm 1, \pm 1), (\pm 3, \pm 1), (\pm 1, \pm 3), (\pm 3, \pm 3)$ and the 64-velocity model consisting of all velocities with components in $\{\pm 1, \pm 3\}$ for $d = 3$

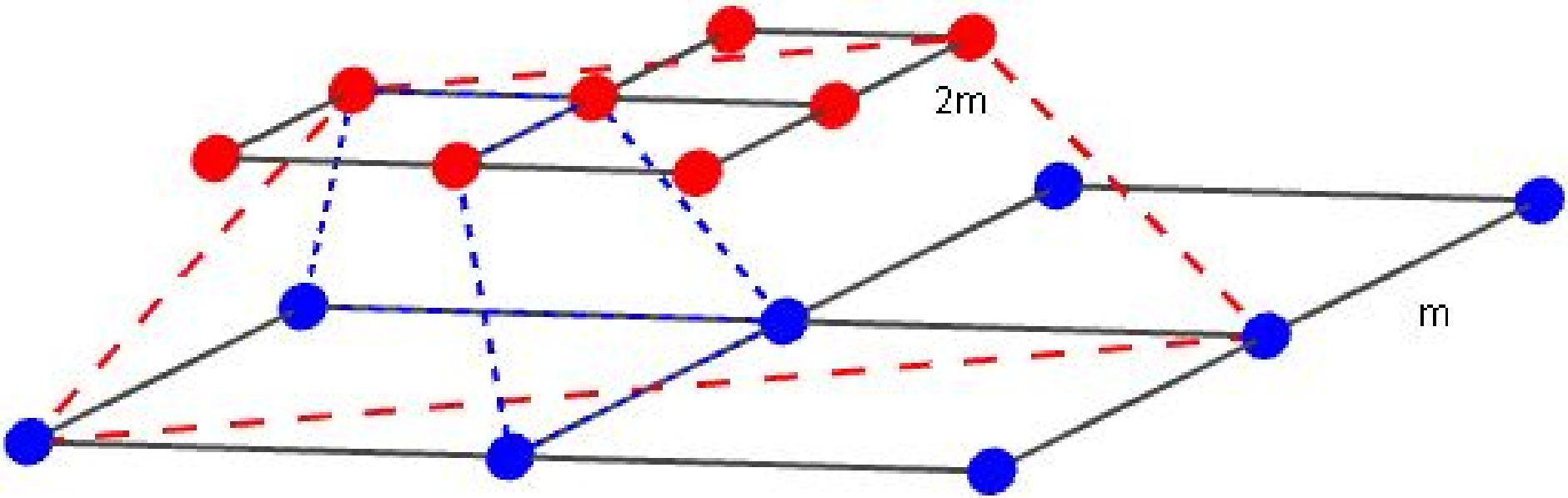
We let

$$V^{\alpha_i} = \frac{h}{m_{\alpha_i}} V, \; i = 1, ..., s,$$

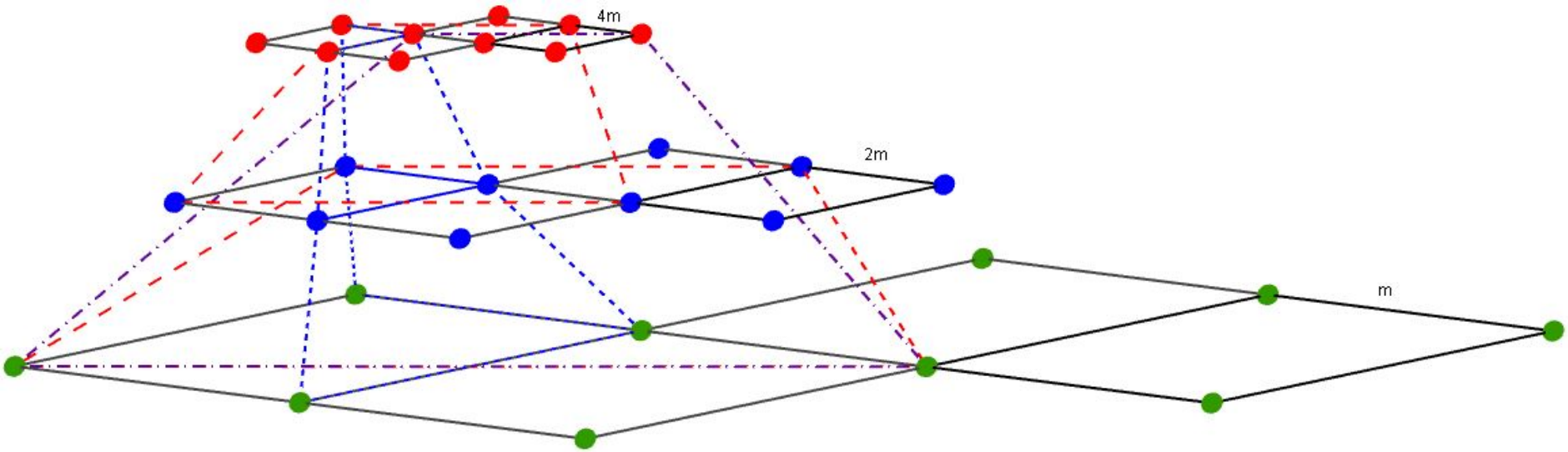
for some positive number $h > 0$

Starting models only allow mass ratio 2

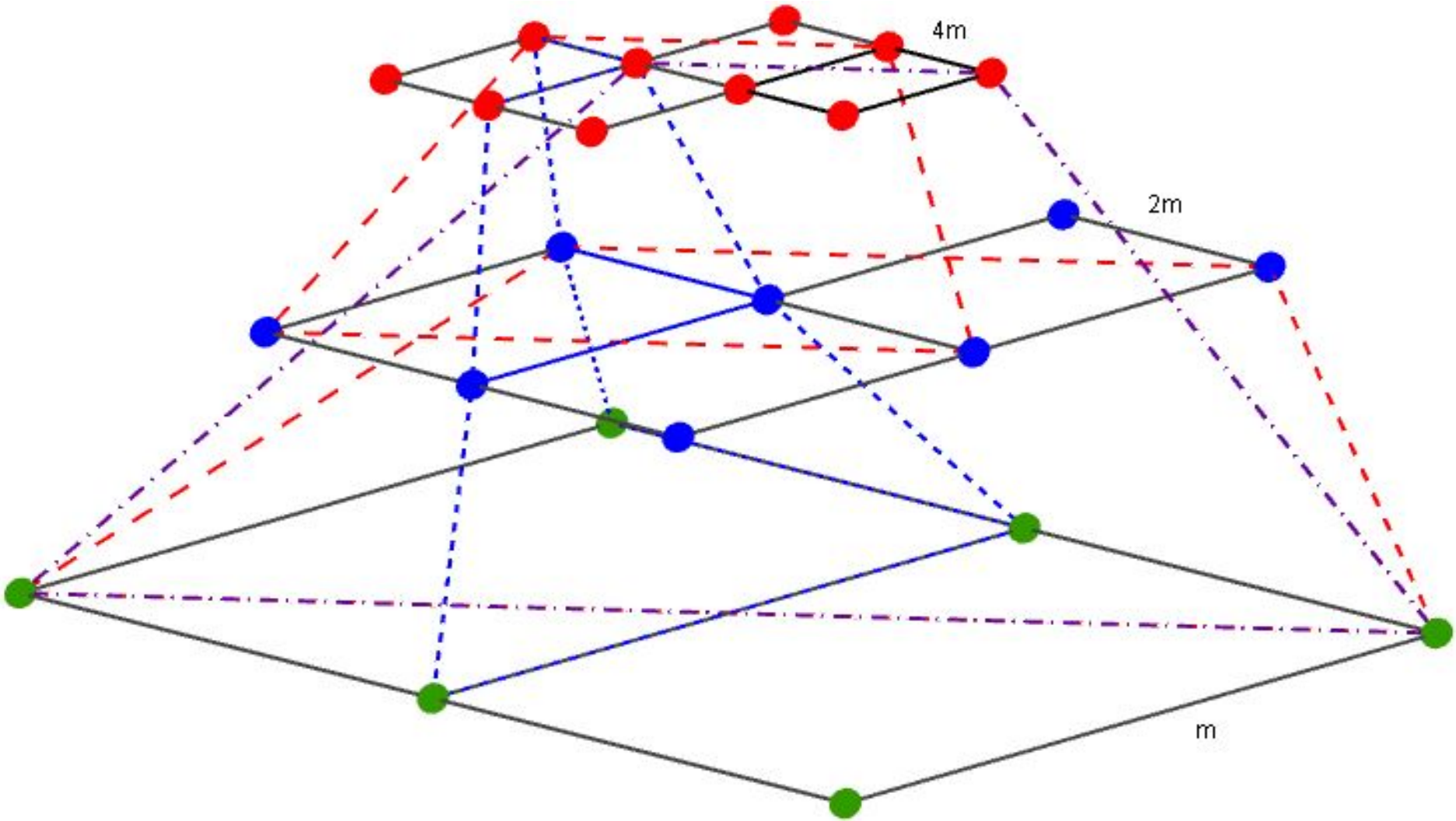
Mixture (mass ratio = 2)



Mixture (mass ratios = 2, 2, 4)



Mixture (mass ratios = 2, 2, 4)



DVMs with internal energies

Assume now that we have s different internal energies $E^1, ..., E^s$. For each internal energy E^i we fix a set of velocity vectors $V_i = \left\{\xi_1^i, ..., \xi_{n_i}^i\right\} \subset \mathbb{R}^d$. We obtain a set of $N = n_1 + ... + n_s$ pairs (each pair being composed of a velocity vector and an internal energy)

$$\begin{aligned}\mathcal{P} &= \left\{\left(\xi_1^1, E^1\right), ..., \left(\xi_{n_1}^1, E^1\right), ..., \left(\xi_1^s, E^s\right), ..., \left(\xi_{n_s}^s, E^s\right)\right\} \\ &= \left\{\left(\mathbf{p}_1, E_1\right), ..., \left(\mathbf{p}_N, E_N\right)\right\}, \text{ with } N = n_1 + ... + n_s.\end{aligned}$$

We might need to rescale the distribution functions

$$f_r' = \frac{f_r}{g_i} \text{ if } E_r = E^i, r = 1, ..., N \text{ for some numbers } g_1, ..., g_s$$

Collision coefficients (after possible rescaling):

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0$$

with equality unless (preservation of momentum and total energy)

$\mathbf{p}_i + \mathbf{p}_j = \mathbf{p}_k + \mathbf{p}_l$ and $m\left|\mathbf{p}_i\right|^2 + m\left|\mathbf{p}_j\right|^2 + E_i + E_j = m\left|\mathbf{p}_k\right|^2 + m\left|\mathbf{p}_l\right|^2 + E_k + E_l$

$d+2$ collision invariants for normal DVMs:

$$\phi = \phi(\mathbf{p}) = a + \mathbf{b} \cdot \mathbf{p} + c(m\left|\mathbf{p}\right|^2 + E(\mathbf{p}))$$

constant $a, c \in \mathbb{R}$, $\mathbf{b} \in \mathbb{R}^d$

Maxwellians:

$$M = e^{\phi}$$

Algorithm for construction of supernormal DVMs for polyatomic molecules

A supernormal DVM for polyatomic molecules is a DVM such that the restriction to each collection of velocity sets is normal

Algorithm:
1) Choose a set of velocities V_1 such that it corresponds to a normal model for single species
2) Iteration step. For $i = 2, \ldots, s$
Choose a normal set of velocities V_i such that, it together with each of $V_1, \ldots, V_{i-1}$ corresponds to a normal model

In each case we need to have $d + 2$ linearly independent (also with respect to the collisions inside the two energy levels) collisions between the two levels of internal energies

Example- construction of normal models with internal energies

We assume that we have the internal energies $E^r = rE, r = 1, \ldots, s$

We start with a set of velocities V, which includes the six velocities $(\pm 1, \pm 1), (3, \pm 1)$ for $d = 2$ or the ten velocities $(\pm 1, \pm 1, \pm 1), (3, \pm 1, 1)$ for $d = 3$, and is normal

Extensions to larger normal models (of any finite size) by one-extensions

The smallest "symmetric" normal models including the minimal models are the 12-velocity model $(\pm 1, \pm 1), (\pm 3, \pm 1), (\pm 1, \pm 3)$ for $d = 2$ and the 32-velocity model $(\pm 1, \pm 1, \pm 1), (\pm 3, \pm 1, \pm 1), (\pm 1, \pm 3, \pm 1), (\pm 1, \pm 1, \pm 3)$ for $d = 3$

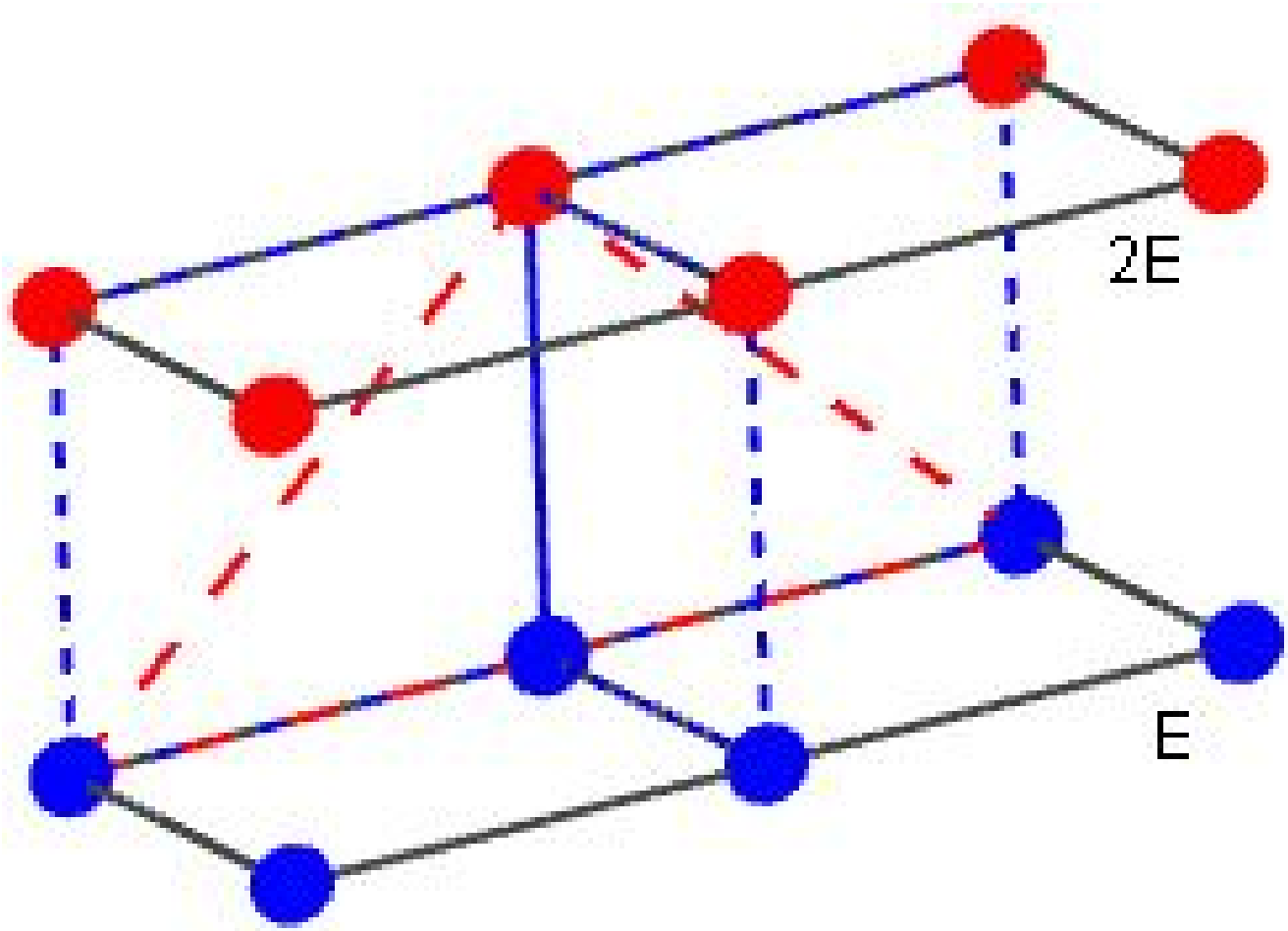
We let

$$V_i = \frac{E}{4m} V, \; i = 1, ..., s,$$

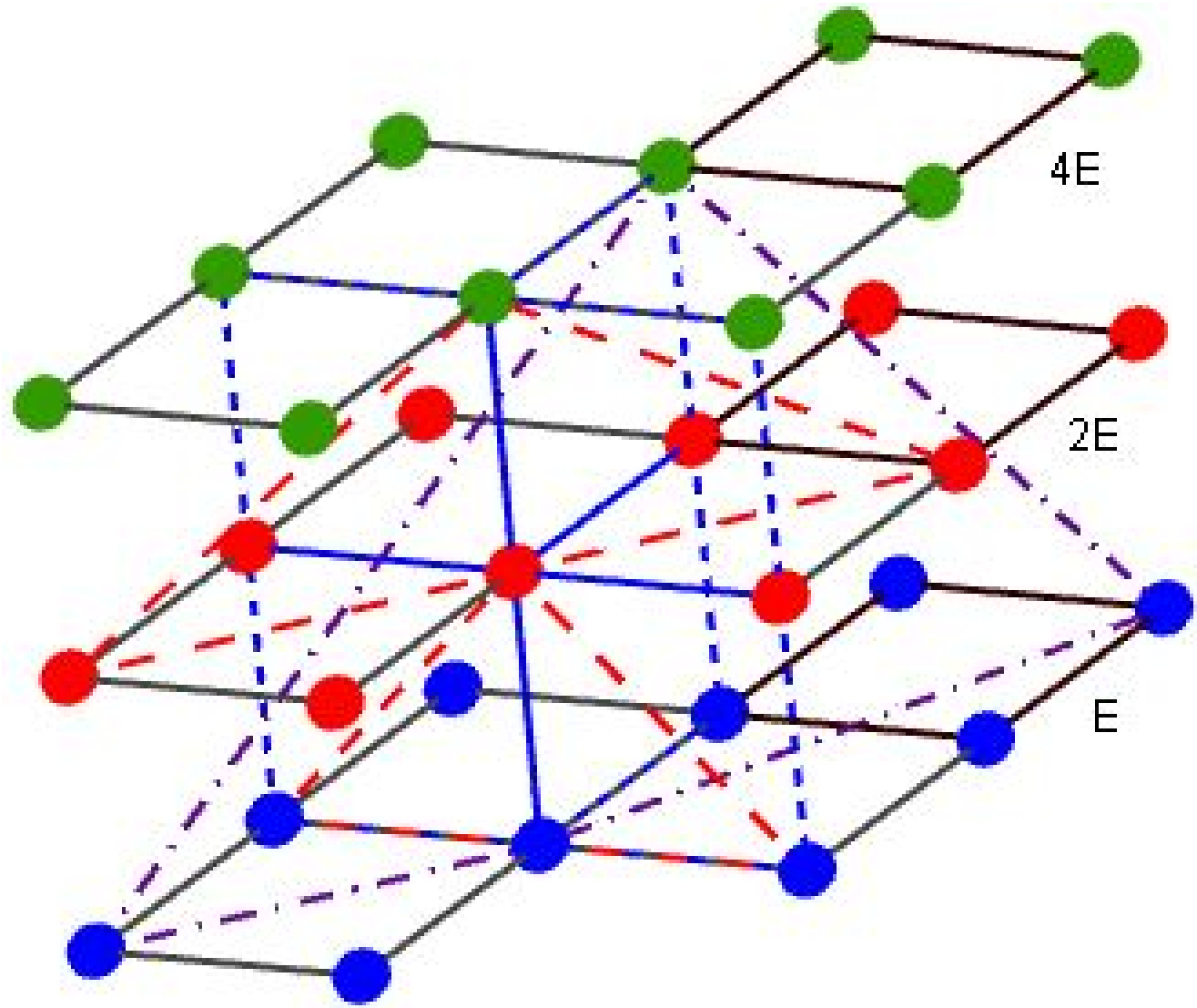
where m is the mass

For minimal models maximal total change of energies under a collision is E

Polyatomic molecules (internal energies E and $2E$)



Polyatomic molecules (internal energies E, $2E$, and $4E$)



Bimolecular chemical reactions

We can combine mixtures with internal energies in an obvious way
Possible extension to chemical reactions by adding reactive collisions

Example

Four species $A^1, ..., A^4$, with masses $m_1, ..., m_4$ and internal energies $E_1, ..., E_4$, respectively. Elastic collisions (as a mixture) plus a reactive interaction

$$A^1 + A^2 \rightleftarrows A^3 + A^4$$

if

$$\begin{aligned} m_1\xi^1 + m_2\xi^2 &= m_3\xi^3 + m_4\xi^4 \text{ and} \\ m_1\left|\xi^1\right|^2 + m_2\left|\xi^2\right|^2 + E_1 + E_2 &= m_3\left|\xi^3\right|^2 + m_4\left|\xi^4\right|^2 + E_3 + E_4 \end{aligned}$$

(implying one less collision invariant) - Spiga, Groppi, Bisi ...

Discrete Nordheim-Boltzmann Equation

(Continuous: Nordheim 1928, Uehling-Uhlenbeck 1933)

The discrete Nordheim-Boltzmann equation (or Uehling-Uhlenbeck equation) reads

$$\frac{\partial F_i}{\partial t} + \mathbf{p}_i \cdot \nabla_{\mathbf{x}} F_i = Q_i^{\varepsilon}(F),\ i = 1, ..., N,$$

where $\mathcal{P} = \{\mathbf{p}_1, ..., \mathbf{p}_N\} \subset \mathbb{R}^d$ is a finite set and $F = (F_1, ..., F_N)$, $F_i = F_i(x, t) = F(x, t, \mathbf{p}_i)$, is the distribution function

Collision operator:

$$Q_i^{\varepsilon}(F) = \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl}(F_k F_l (1+\varepsilon F_i)(1+\varepsilon F_j) - F_i F_j (1+\varepsilon F_k)(1+\varepsilon F_l))$$

$$Q^{\varepsilon} = (Q_1^{\varepsilon}, ..., Q_n^{\varepsilon})$$

Collision coefficients:

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0$$

with equality unless

$$\mathbf{p}_i + \mathbf{p}_j = \mathbf{p}_k + \mathbf{p}_l \text{ and } |\mathbf{p}_i|^2 + |\mathbf{p}_j|^2 = |\mathbf{p}_k|^2 + |\mathbf{p}_l|^2$$

$\varepsilon = 0$: classical discrete Boltzmann equation (DBE)
$\varepsilon = 1$: bosons
$\varepsilon = -1$: fermions

Collision invariants:

$$\phi_i + \phi_j = \phi_k + \phi_l \text{ if } \Gamma_{ij}^{kl} \neq 0$$

Normal models (only with physical collision invariants):
(Constructions: Bobylev and Cercignani 1999 (JSP), Bobylev and Vinerean 2008 (JSP) etc.):

$$\phi = (\phi_1, ..., \phi_N),\ \phi_i = a + \mathbf{b} \cdot \mathbf{p}_i + c\,|\mathbf{p}_i|^2$$

$$\text{constant } a, c \in \mathbb{R},\ \mathbf{b} \in \mathbb{R}^d$$

Maxwellian distributions/Maxwellians:

$$M = (M_1, ..., M_N),\ M_i = e^{\phi_i} = Ke^{\mathbf{b}\cdot\mathbf{p}_i + c|\mathbf{p}_i|^2},\ \text{with } K = e^a > 0$$

Equilibrium distributions: Planckian distributions/Planckians:

$$P = \frac{M}{1 - \varepsilon M}$$

where M is a Maxwellian

$$Q^\varepsilon(F) = 0 \text{ and } F_i \geq 0 \text{ if and only if } F = P$$

Linearized collision operator

Denote:

$$F_i = P_i + \sqrt{P_i(1+\varepsilon P_i)} f_i,\ i = 1, ..., N$$

New system (of PDEs):

$$\frac{\partial f}{\partial t} + \mathbf{p} \cdot \nabla_{\mathbf{x}} f + Lf = S(f)$$

The linearized operator ($n \times n$-matrix) L is symmetric and positive semi-definite

$$\langle g, Lh \rangle = \langle Lg, h \rangle \text{ and } \langle h, Lh \rangle \geq 0,$$

$\langle \cdot, \cdot \rangle$ - Euclidean scalar product
Kernel of L (for normal model):

$$N(L) = \left\{ \sqrt{P(1+\varepsilon P)}\phi,\ \phi \text{ collision invariant} \right\} =$$

$$\text{span} \left\{ \sqrt{P(1+\varepsilon P)},\ \sqrt{P(1+\varepsilon P)}\mathbf{p},\ \sqrt{P(1+\varepsilon P)}\,|\mathbf{p}|^2 \right\}$$

Nonlinear part

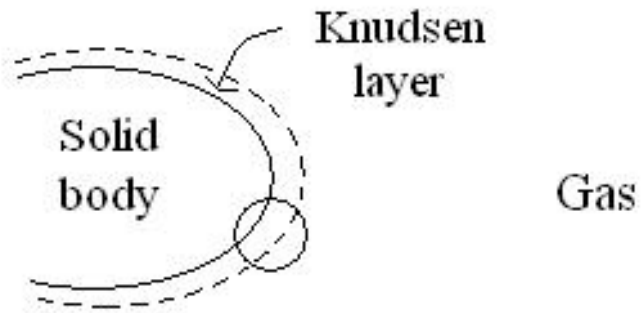
The nonlinear part is orthogonal to the kernel of L:

$$\langle S(f), y\rangle = 0 \text{ if } y \in N(L)$$

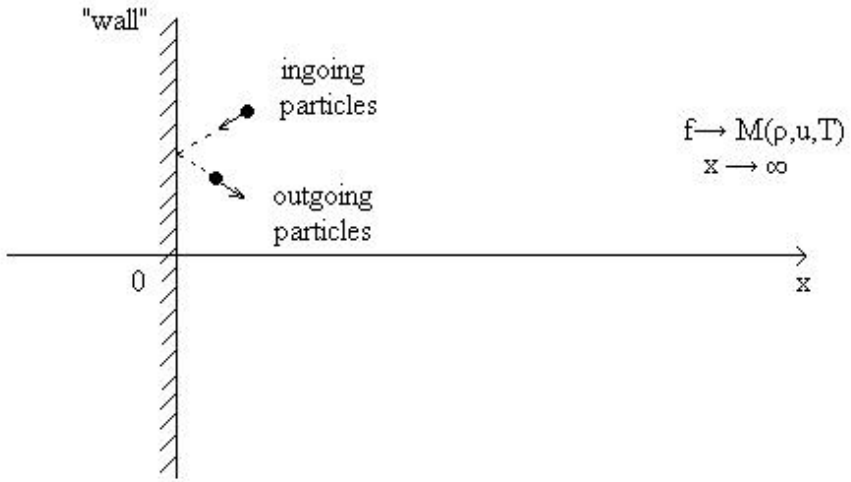
and, for some konstant $\widetilde{K}$:

$$|S(g) - S(h)| \leq \widetilde{K}(|g| + |h|)(1 + |g| + |h|)\,|g - h|$$

Half-space problem of evaporation and condensation for the Boltzmann equation



$$\xi^1 \frac{\partial f}{\partial x} = Q(f, f)$$



Non-linear half-space problems

Boltzmann equation:
Sone 2002, 2007
Coron, Golse and Sulem 1988 (CPAM)
Ukai, Yang and Yu 2003 (CMP),
Bardos, Golse and Sone 2006 (JSP)
Golse 2008 (BIMAS)
Discrete Boltzmann equation:
Ukai 1998
Kawashima and Nishibata 1999, 2000 (CMP),
Babovsky 1999 (JCAM)
NB 2010 (KRM)
Binary mixtures of two vapors BE:
Takata and Golse 2007 (EJMBF)
Yang and Bardos 2012 (CPDE)
NB 2012 (KRM) - DBE
Quantum BE:
NB (in preparation)

$$\langle x, Lx \rangle \geq 0, \ \det B \neq 0 \text{ and } \dim(N(L)) = q \geq 1$$

The $q \times q$ matrix K with entries

$$k_{ij} = \langle g_i, Bg_j \rangle ,$$

has the same number of positive (k^+), negative (k^-) and zero ($l = q - k^+ - k^-$) eigenvalues, independently of the choice of the basis $\{g_1, ..., g_q\}$ of $N(L)$

$$B = \begin{pmatrix} B_+ & 0 \\ 0 & -B_- \end{pmatrix},$$

$$B_+ = \text{diag}\left(p_1^1, ..., p_{n^+}^1\right) \text{ and } B_- = -\text{diag}\left(p_{n^++1}^1, ..., p_N^1\right), \text{ with}$$
$$p_1^1, ..., p_{n^+}^1 > 0 \text{ and } p_{n^++1}^1, ..., p_N^1 < 0$$

Projections:

$$R_+ s = s^+ = (s_1, ..., s_{n^+}) \text{ and } R_- s = s^- = (s_{n^++1}, ..., s_N)$$

for $s = (s_1, ..., s_N)$

Jordan normal form of $B^{-1}L$:

$$\begin{pmatrix}
\lambda_1 & & & & & & & & & \\
& \ddots & & & & & & & & \\
& & \lambda_q & & & & & & & \\
& & & 0 & & & & & & \\
& & & & \ddots & & & & & \\
& & & & & 0 & & & & \\
& & & & & & 0 & 1 & & \\
& & & & & & 0 & 0 & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}
\end{pmatrix}$$

l blocks of the type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$l = 0$ - non-degenerate case

$l \geq 1$ - degenerate case

Weakly nonlinear half-space problem

System:

$$\begin{cases} B\dfrac{df}{dx} + Lf = S(f) \\ f^{+}(0) = h_0 \\ f(x) \to 0 \text{ as } x \to \infty \end{cases}$$

Nonlinear part:

$$S\left(f\right) \in N(L)^{\perp}$$

and

$$\left|S\left(g\right) - S\left(h\right)\right| \leq \widetilde{K} G(\left|g\right|, \left|h\right|) \left|g - h\right|$$

where $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function with positive partial derivatives, such that $G(0,0) = 0$

Existence result for nonlinear boundary layers

Theorem

Suppose that $\langle h_0, B_+ h_0 \rangle$ is sufficiently small. Then with $k^+ + l$ conditions on h_0, the system has a locally unique solution.

- our results can be extended to yield also for singular matrices B (NB 2010 (KRM)), if

$$N(L) \cap N(B) = \{0\}$$

- we can consider more general boundary conditions:

$$R_+ f(0) = C R_- f(0) + h_0$$

where C is a given $n^+ \times n^-$ matrix and $h_0 \in \mathbb{R}^{n^+}$ (with some restrictions on the matrix C)

Symmetric models

Change of variables:

$$\mathbf{p} \to \mathbf{p} + \mathbf{p}_0 \ (u = p_0^1)$$

Matrix K with entries

$$k_{ij} = \langle g_i, (B + uI) g_j \rangle \,,$$

$\{g_1, ..., g_p\}$ basis of $N(L)$

Nordheim-Boltzmann equation:

		$u = -u_+$		$u = 0$		$u = u_+$	
k^+	0	0	1	1	$d+1$	$d+1$	$d+2$
k^-	$d+2$	$d+1$	$d+1$	1	1	0	0
l	0	1	0	d	0	1	0

$$u_+ = \sqrt{\frac{\chi_1\chi_4^2 + \chi_2^2\chi_5 - 2\chi_2\chi_3\chi_4}{\chi_2(\chi_1\chi_5 - \chi_3^2)}}, \text{ where}$$

$$\chi_1 = \langle \phi_1, \phi_1 \rangle, \chi_2 = \langle \phi_2, \phi_2 \rangle, \chi_3 = \langle \phi_1, \phi_{d+2} \rangle, \chi_4 = \langle \phi_2, \phi_{d+2} \rangle_B$$
$$\text{and } \chi_5 = \langle \phi_{d+2}, \phi_{d+2} \rangle.$$

and, with $R = P(1+P)$,

$$\begin{cases} \phi_1 = R^{1/2} = R^{1/2} \cdot (1, ..., 1) \\ \phi_2 = R^{1/2} p^1 = R^{1/2} \cdot (p_1^1, ..., p_N^1, -p_1^1, ..., -p_N^1) \\ \phi_3 = R^{1/2} |\mathbf{p}|^2 = R^{1/2} \cdot (|\mathbf{p}_1|^2, ..., |\mathbf{p}_N|^2, |\mathbf{p}_1|^2, ..., |\mathbf{p}_N|^2) \\ \phi_{i+2} = R^{1/2} p^i = R^{1/2} \cdot (p_1^i, ..., p_N^i, -p_1^i, ..., -p_N^i), \ i = 2, ..., d, \end{cases}$$

BE (Coron, Golse, Sulem 1988 (CPAM), $d = 3$): $u_+ = \sqrt{\frac{5T}{3m}}$

if

$$M = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{m|\mathbf{p}|^2}{2T}}$$

Fermions $(d = 3)$, $P = \dfrac{1}{e^{\frac{|\mathbf{p}|^2}{T}} + 1}$:

$$u_+ = \sqrt{\frac{\eta\left(5/2\right)}{\eta\left(3/2\right)}}\sqrt{\frac{5T}{6}}$$

where η is the Dirichlet eta-function/alternating zeta-function

Bosons $(d = 3)$, $P = \dfrac{1}{e^{\frac{|\mathbf{p}|^2}{T}} - 1}$:

$$u_+ = \sqrt{\frac{I_6}{I_4}}\sqrt{\frac{T}{3}} \to \sqrt{\frac{\zeta\left(5/2\right)}{\zeta\left(3/2\right)}}\sqrt{\frac{5T}{6}} \text{ as } \lambda \to 0$$

where ζ is the zeta-function and $I_n = \int_\lambda^\infty r^n \dfrac{e^{r^2}}{\left(e^{r^2} - 1\right)^2}\, dr$, $\lambda > 0$

For mixtures: $d \to d + s - 1$

		$u = u_-$		$u = 0$		$u = u_+$	
k^+	0	0	1	1	$s + d$	$s + d$	$s + d + 1$
k^-	$s + d + 1$	$s + d$	$s + d$	1	1	0	0
l	0	1	0	$s + d - 1$	0	1	0

$u_0 = 0$ and $u_\pm = \pm \sqrt{\dfrac{\mathcal{X}}{\chi_2 \left(\sum_{i=1}^{s} \dfrac{(\chi_3^{\alpha_i})^2}{\chi_1^{\alpha_i}} - \chi_5 \right)}}$, with

$$\mathcal{X} = \chi_4^2 + \chi_5 \sum_{i=1}^{s} \frac{(\chi_2^{\alpha_i})^2}{\chi_1^{\alpha_i}} - 2\chi_4 \sum_{i=1}^{s} \frac{\chi_2^{\alpha_i} \chi_3^{\alpha_i}}{\chi_1^{\alpha_i}} - \sum_{i=1}^{s} \frac{(\chi_2^{\alpha_i} \chi_3^{\alpha_j} - \chi_2^{\alpha_j} \chi_3^{\alpha_i})^2}{\chi_1^{\alpha_i} \chi_1^{\alpha_j}}.$$

where $\chi_1^{\alpha_i} = \langle \phi_1^{\alpha_i}, \phi_1^{\alpha_i} \rangle$, $\chi_2^{\alpha_i} = \langle \phi_1^{\alpha_i}, \phi_2 \rangle_{B_{\alpha_s}} = m_{\alpha_s} \langle \phi_2^{\alpha_i}, \phi_2^{\alpha_i} \rangle$,
$\chi_2 = \langle \phi_2, \phi_2 \rangle$, $\chi_3^{\alpha_i} = \langle \phi_1^{\alpha_i}, \phi_3 \rangle = m_{\alpha_s} \langle \phi_1^{\alpha_i}, \phi_3^{\alpha_i} \rangle$, $\chi_4 = \langle \phi_2, \phi_3 \rangle_B$, and
$\chi_5 = \langle \phi_3, \phi_3 \rangle$
and, with $R = P(1+P)$,

$$\begin{cases} \phi_1^{\alpha_i} = R^{1/2} \cdot (\underbrace{0, ..., 0}_{2\sum_{j=1}^{i-1} N_{\alpha_j}}, \underbrace{1, ..., 1}_{2N_{\alpha_i}}, \underbrace{0, ..., 0}_{2\sum_{j=i+1}^{s} N_{\alpha_j}}), \ i = 1, ..., s, \\ \phi_2 = R^{1/2} \cdot (m_{\alpha_1} \phi_2^{\alpha_1}, ..., m_{\alpha_s} \phi_2^{\alpha_s}), \text{ with } \phi_2^{\alpha} = (\xi_1^{\alpha,1}, ..., \xi_{N_\alpha}^{\alpha,1}, -\xi_1^{\alpha,1}, ..., -\xi_{} \\ \phi_3 = R^{1/2} \cdot (m_{\alpha_1} \phi_3^{\alpha_1}, ..., m_{\alpha_s} \phi_3^{\alpha_s}), \text{ with } \phi_3^{\alpha} = (|\xi_1^{\alpha}|^2, ..., |\xi_{N_\alpha}^{\alpha}|^2, |\xi_1^{\alpha}|^2, ..., \\ \phi_{2+i} = R^{1/2} \cdot (m_{\alpha_1} \phi_{2+i}^{\alpha_1}, ..., m_{\alpha_s} \phi_{2+i}^{\alpha_s}), \text{with } \phi_{2+i}^{\alpha} = (\xi_1^{\alpha,i}, ..., \xi_{2N_\alpha}^{\alpha,i}), i = 2, . \end{cases}$$

BE ($d = 3$):

$$u_0 = 0 \text{ and } u_\pm = \pm\sqrt{\frac{\sum_{i=1}^{s} n_{\alpha_i}}{\sum_{i=1}^{s} m_{\alpha_i} n_{\alpha_i}}}\sqrt{\frac{5T}{3}}.$$

if

$$M = (M_{\alpha_1}, ..., M_{\alpha_s}), \text{ with } M_{\alpha_i} = \frac{n_{\alpha_i}}{(2\pi T)^{3/2}} e^{-\frac{m_{\alpha_i}|\xi|^2}{2T}}.$$

THANK YOU!