Factorizations of weighted function spaces

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1. The theoretical framework

Let Y be a space. We say that Y is factorized by X and Z if $Y = X \cdot Z$.

To factorize Y with respect to X means to find Z such that $X \cdot Z = Y$. (if such a Z exists!)

If X is a (quasi)-normed space, the inequality expressing the boundeness of a linear (or sublinear) operator T on X can be interpreted as the inclusion between the spaces X and Y, where

$$Y = \{x : \|x\|_Y = \|Tx\|_X < \infty\}$$

.

One way of improving the operator inequality is by substituting the space X by a larger space $X \subset \tilde{X} \subset Y$ which preserves the constants of the embeddings.
We would have an enhancement of the inequality becouse the inequality holds now for a larger collection $\tilde{X}$. An $\tilde{X}$ satisfying the above properties is $X \cdot Z$, where Z is the space those elements z such that $x \cdot z$ are in Y (the sets of multipliers from X to Y). If Z can be find, it is possible to show, in some cases that also
$Y \subset X \cdot Z$.
The first inclusion gives an optimal form of the operator inequality and the second gives a renorming of the space Y.

2. Factorizations of classical spaces of sequences

(Grahame Bennet)Let $p > 1$. The classical Hardy inequality

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{1/p} \leq \frac{p}{p-1}\left(\sum_{n=1}^{\infty}x_n^p\right)^{1/p}$$

and can be roughfly interpreted as

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty$$

whenever $\sum_{n=1}^{\infty}|x_n|^p < \infty$, or as inclusion between the sequence spaces $l^p \subset \text{ces(p)}$. We seek to replace l^p, by a larger space, such as first of all, Hardy inequality remains visible. We want to find the set Z of multipliers from l^p into ces(p), namely those sequences z, with the property that $xz \in \text{ces(p)}$ whenever $x \in l^p$. Clearly, $l^p \cdot Z \subset \text{ces(p)}$. In this case, surprisingly, Z could be described in very simple terms and the inclusion turns up to be an identity.

Theorem: Let $p > 1$. A sequence $y \in \text{ces(p)}$ if and only if it admits a factorization $y = x \cdot z$, with $x \in l^p$ and $z \in \mathrm{g}(p')$, where

$$\mathrm{g}(p) = \{z : |z_1|^p + \cdots + |z_n|^p = O(n)\}$$

or

$$\text{ces(p)} = l^p \cdot \mathrm{g}(p')$$

$\mathrm{g}(p)$ are Banach spaces, with $\|x\|_{\mathrm{g}(p)} = \sup \left(\frac{\sum_k^n |x_k|^p}{n} \right)^{1/p}$. ($\mathrm{g}(p)$ is a "big" space since it contains unbounded sequences!)

Necessity: Let $y \in \operatorname{ces}(p)$. Define

$$x_n = \left(|y_n| \sum_{k=n}^{\infty} \frac{1}{k^p} \left(\sum_{j=1}^{k} |y_n| \right)^{p-1} \right)^{1/p} \quad \operatorname{sgn}(y_n)$$

and

$$z_n = |y_n|^{1/p'} \left(\sum_{k=n}^{\infty} \frac{1}{k^p} \left(\sum_{j=1}^{k} |y_n| \right)^{p-1} \right)^{-1/p} \quad .$$

Observ that $y = x \cdot z$. One can proove that $\|x\|_p = \|y\|_{\operatorname{ces}(p)}$ and $\|z\|_{\mathrm{g}(p')} \leq (p-1)^{1/p}$, hence beside the inclusion, this gives

$$(p-1)^{-1/p} !y!_p \leq \|y\|_{\operatorname{ces(p)}}$$

where $!y!_p = \inf \|x\|_p \|z\|_{\mathrm{g}_{p'}}$, the infimum being taken over all factorizations $y = x \cdot z$.

Sufficiency:
Suppose now, that y admits a factorization $y = x \cdot z$, with $x \in l^p$ and $z \in \mathrm{g}(p')$. it is proved (Hölder, Hardy inequalites, etc) that $y \in \mathrm{ces}(p)$ and moreover

$$\|y\|_{\mathrm{ces(p)}} \leq p' !y!_p$$

where $!y!_p = \inf \|x\|_p \|z\|_{\mathrm{g}'_p}$, the infimum being taken over all factorizations $y = x \cdot z.$, $x \in l^p$, $z \in \mathrm{g}(p')$.

Remarks:

- 1. The theorem contains Hardy inequality as a special case. Let $y \in l^p$. $x = y \cdot 1$. The theorem implies $y \in \text{ces}(p)$.
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Consequences:

- 1. The factorization gives a better understanding of the structure of Cesaro spaces; $!y!_p$ is a actually a new norm on this space.

- 2. The factorization is used to give an easier explicit description for the dual of ces(p) and show that ces(p) is reflexive. The problem of finding explicitly the dual of ces(p) was posed by The Duch Mathematical Society, in 1971, as a "prijsvrage" and the solution given in 1974, by A.A. Jagers was very complicated. The dual of ces(p) is $d(p')$. What is $d(p)$?

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- 3. ces(p) is not isomorphic with any l^q nor any Orlicz space.

- 4. ces(p) have a property which is not shared by any other space of sequences, namely $x \in$ ces(p) if and only if $y_n = \frac{x_1 + \cdots + x_n}{n}$ belongs to ces(p).

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- 1. The Copson space $\mathrm{cop}(p) := \{x : \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|x_k|}{k} \right)^p < \infty\}$ has the same factorization as the Cesaro space $\mathrm{ces}(p)$., (the constants only are different.)
- 2. The factorization of Copson spaces was proved by a completely different method than that of of Cesaro. These two factorizations show indirectly that $\mathrm{cop}(p) = \mathrm{ces}(p)$ as spaces of sequences; the two best constants of the inclusions are being given, ($1 < p < 2$, $p \geq 2$; the other cases, still open question; solved by V. Kolyada for integrals.)

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3. Motivations and factorizations of weighted Cesaro space of functions

The case of spaces of functions is named in the book. Many of the results are just straightforward, except the factorization
$l^p = \mathrm{g}(p) \cdot \mathrm{d}(p)$.
We are interested in factorizing the weighted Cesaro and Copson spaces of functions, since they are a "hot topic", the characterization of weights for which Hardy inequality holds is known and the factorization made by C. Carton-Lebrun and H. P. Heinig (2003, J. Math. Anal. Appl.) is not satisfactory!

Let $p > 1$ and $Hf(t) = \frac{1}{t}\int_0^t f(x)dx$. it is known (B. Muckenhoupt, 1972, Studia Math.) that a weight w satisfies the condition

$$M_p : \left(\int_t^\infty \frac{w(x)}{x^p}dx\right)^{1/p}\left(\int_0^t w^{1-p'}(x)dx\right)^{1/p'} \leq C$$

if and only if H is bounded on the weighted Lebesgue space $L^p(w)$. Moreover, if $[w]_{M_p}$ denotes the least constant for which the condition M_p is satisfied it is known that

$$[w]_{M_p} \leq \|H\| \leq p^{1/p}{p'}^{1/p'}[w]_{M_p}.$$

The weighted Cesaro space, $\mathrm{Ces}_p(w)$ is defined as the space of measurable functions, sucha that $\|H(|f|)\|_{p,w} < \infty$.

The factorization of the Cesaro space given by C. Carton-Lebrun and H. P. Heinig:

$$\mathrm{Ces}_p(w) = L_p(v) \cdot G_p(v, w)$$

where
$G_p(v, w) = \{g : \sup_{t>0} \left(\int_t^\infty \frac{w(s)}{s^p} ds\right)^{1/p} \left(\int_0^t v^{1-p'} g^{p'} ds\right)^{1/p'} \}$.
Moreover

$$\inf \|f\|_{p,v} \|g\|_{v,w} \le \|h\|_{\mathrm{Ces}_p(w)} \le 2p^{1/p} {p'}^{1/p'} \inf \|f\|_{p,v} \|g\|_{v,w}$$

where the infimum is taken over all possible factorizations of $h = f \cdot g$. Neither the old Hardy inequlity is visible nor the factorization of the classical spaces (without weights.)

The factorization of the Cesaro space given by us: If $w \in M_p$ and m_p, then

$$\mathrm{Ces}_p(w) = L_p(w) \cdot G_{p'}(w^{1-p'})$$

where $G_p(w) = \{g : \sup_{t>0} \left(\frac{1}{\int_0^t w} \int_0^t g^p w ds \right)^{1/p} < \infty\}$. Moreover

$$[w]_{m_p} \inf \|f\|_{p,w} \|g\|_{G_{p'}(w^{1-p'})} \leq \|h\|_{\mathrm{Ces}_p(w)} \leq$$

$$p^{1/p} {p'}^{1/p'} [w]_{M_p} \inf \|f\|_{p,w} \|g\|_{G_{p'}(w^{1-p'})}$$

where the infimum is taken over all possible factorizations of $h = f \cdot g$.

The best known form of Hardy inequality on weighted Lebesgue spaces is contained above. The constant in the left hand-side of the inequality is optimal. The case of power weights is interesting. If the weight $w = 1$, we get the old results. The case $p = 1$ is proved.

By the same method we can factorize the Copson spaces. The factorization of the Copson spaces given by us: If $w \in M_p^*$ and m_p^*, then

$$\mathrm{Cop}_p(w) = L_p(w) \cdot \tilde{G}_{p'}(w^{1-p'}/t^{p'})$$

where $\tilde{G}_p(w) = \{g : \sup_{t>0} \left(\frac{1}{\int_t^\infty w} \int_t^\infty g^p w ds \right)^{1/p} < \infty\}$.
Moreover

$$[w]_{m_p^*} \inf \|f\|_{p,w} \|g\|_{\tilde{G}_{p'}(w^{1-p'})} \leq \|h\|_{\mathrm{Cop}_p(w)} \leq$$

$$p^{1/p} {p'}^{1/p'} [w]_{M_p^*} \inf \|f\|_{p,w} \|g\|_{\tilde{G}_{p'}(w^{1-p'})}$$

where the infimum is taken over all possible factorizations of $h = f \cdot g$.

It is known that $\mathrm{Ces}_p(w) = \mathrm{Cop}_p(w)$ if $w \in M_p \bigcup M_p^*$. This mean that they must have the same factorization, but we are not able to prove that $\tilde{G}_{p'}(w^{1-p'}/t^{p'} = G_{p'}(w^{1-p'}/t^{p'}$ by keeping track of the constants between the norms.
We also have $L^(w) = D^p(w) \cdot G^p(w)$ as in the case of sequences.

Open questions

-dual of the weighted Cesaro space, show that $!h!$ is a norm, etc.
-Can we use the technique of factorization to factorize the Cesaro-Hardy space of analytic functions on the unit disc? These would give a better understanding of the boundedness of Cesaro averaging operator $Cf(z) = \frac{1}{z}\int_0^z \frac{f(\xi)}{1-\xi} d\xi$ on Hardy spaces of analytic functions. If $f(z) = \sum_n a_n z^k$ and $g(z) = \sum_n (\frac{1}{n+1}\sum_{k=0}^n a_k) z^n$, then $g = C(f)$